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COSINE FAMILIES AND DAMPED SECOND ORDER DIFFERENTIAL EQUATIONS.(U)

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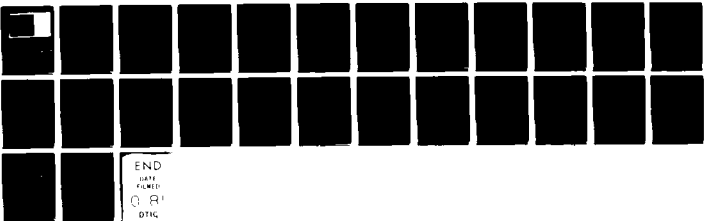
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DIFFERENTIAL EQUATIONS

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ABSTRACT

Consider the abstract differential equation

$$(1) \quad u''(t) + 2Bu'(t) = Au(t) + F(u(t)), \quad t \in \mathbb{R}, \quad u(0) = x, \\ u'(0) = y$$

where A and B are densely defined linear operators and F is possibly nonlinear and unbounded. Assuming that $A + B^2$ generates a cosine family $C(t)$ and $-B$ generates a group $T(t)$, there is a variation of constants formula for (1); namely

$$(2) \quad u(t) = T(t)[C(t)x + S(t)(Bx + y) \\ + \int_0^t T(t-s)S(t-s)F(u(s)) \, ds,$$

where $S(t)$ is the sine family associated with $C(t)$. The motivating examples include $w_{tt} + 2h(x)w_t = w_{xx} + f(w, w_x, w_t)$ and $w_{tt} + 2w_{tx} = w_{xx} + f(w, w_x, w_t)$, for $0 < x < \pi$, $t \in \mathbb{R}$, $w(x, 0) = h(x)$, $w_t(x, 0) = g(x)$, and various boundary conditions. We examine the existence of mild solutions and the asymptotic behavior when there is a damping effect introduced by the $2Bu'(t)$ term.

AMS(MOS) Subject Classification: 34G20, 35L15

Key Words: abstract differential equations, strongly continuous cosine family, strongly continuous group

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SIGNIFICANCE AND EXPLANATION

A variation of constants formula is given for certain second order differential equations in a Banach space. The abstract results obtained can be applied to a class of damped semilinear hyperbolic partial differential equations; in particular, the existence and asymptotic behavior of solutions of such equations is examined.

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COSINE FAMILIES AND DAMPED SECOND ORDER DIFFERENTIAL EQUATIONS

James H. Lightbourne, III
Samuel M. Rankin, III

1. INTRODUCTION. Let X be a Banach space and A and B be linear operators on X with domains $D(A)$ and $D(B)$ respectively. F will denote a nonlinear, possibly unbounded map on X . We consider the abstract differential equation:

$$(1.1) \quad u''(t) + 2Bu'(t) = Au(t) + F(u(t)), \quad t \in \mathbb{R}$$

$$u(0) = x, \quad u'(0) = y$$

Essentially under the assumption that $A + B^2$ generates a cosine family $C(t)$, $t \in \mathbb{R}$, of linear operators on X and that $-B$ generates a group $T(t)$, we will establish existence for (1.1) and examine the asymptotic behavior of (1.1) when there is a damping effect introduced by the term $2Bu'(t)$. We will actually consider "mild" solutions of (1.1); i.e., solutions of the variation of constants equation:

$$(1.2) \quad u(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)S(t-s)F(u(s))ds,$$

where $S(t)$ is the sine family associated with $C(t)$.

Two situations to which the abstract theory applies are indicated by the following examples.

$$(1.3) \quad w_{tt}(x,t) + 2b(x)w_t(x,t) = w_{xx}(x,t) + f(w(x,t), w_x(x,t)),$$

$$t \in \mathbb{R}, \quad 0 < x < \pi$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \leq x \leq \pi$$

$$w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}$$

where $b: [0,\pi] \rightarrow \mathbb{R}$ is continuous. Asymptotic behavior for (1.3)

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and $f \equiv 0$ has been considered by Rauch [3]. The second illustrative example is

$$(1.4) \quad w_{tt}(x,t) + 2w_{xt}(x,t) = w_{xx}(x,t) + f(w(x,t), w_x(x,t)),$$

$$t \in \mathbb{R}, \quad 0 < x < \pi$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \leq x \leq \pi$$

$$w(0,t) = w(\pi,t), \quad w_x(0,t) = w_x(\pi,t), \quad t \in \mathbb{R}$$

The preliminaries in section 2 include the known properties of cosine families that we will use and the assumptions on A and B which will be made throughout the paper. Also, in section 2 we establish the relationship between equations (1.1) and (1.2). In section 3 we give existence criteria for equation (1.2) and some global properties of solutions are given in section 4. The examples are discussed in section 5.

2. PRELIMINARIES. Let X be a Banach space with norm $|\cdot|$.

DEFINITION. A one-parameter family $\{C(t): t \in \mathbb{R}\}$ of bounded linear operators on X is called a strongly continuous cosine family provided

- (i) $C(0) = I$, the identity on X ;
- (ii) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathbb{R}$; and
- (iii) for each $x \in X$, $C(\cdot)x: \mathbb{R} \rightarrow X$ is continuous.

Associated with $C(t)$ is the sine family $\{S(t): t \in \mathbb{R}\}$ defined by $S(t)x = \int_0^t C(s)x \, ds$ for $x \in X$. the infinitesimal generator of $C(t)$ is the linear operator $G: D(G) \rightarrow X$ defined by $Gx = C''(0)x$ where

$$D(G) = \{x \in X: C(\cdot)x: \mathbb{R} \rightarrow X \text{ is twice continuously differentiable}\}.$$

We also refer to the set E defined by

$$E = \{x \in X: C(\cdot)x: \mathbb{R} \rightarrow X \text{ is continuously differentiable}\}.$$

The proof of the following proposition as well as a more complete discussion of cosine families may be found in Travis and Webb [4].

PROPOSITION 2.1. *Let $\{C(t): t \in \mathbb{R}\}$ be a strongly continuous cosine family of bounded linear operators on X with generator G . The following properties hold:*

- (i) G is a closed operator on X with domain $D(G)$ dense in X ;
- (ii) if $x \in X$, then $S(t)x \in E$ and $S'(t)x = C(t)x$;
- (iii) if $x \in E$, then $S(t)x \in D(G)$ and $S''(t)x = GS(t)x$;
- (iv) if $x \in E$, then $C'(t)x = GS(t)x$;
- (v) if $x \in D(G)$, then $S(t)x \in D(G)$ and $GS(t)x = S(t)Gx$;
- (vi) if $x \in D(G)$, then $C(t)x \in D(G)$ and $C''(t)x = GC(t)x = C(t)Gx$;
- (vii) $C(t+s) - C(t-s) = 2GS(t)S(s)$, for all $s, t \in \mathbb{R}$;
- (viii) $S(s+t) = S(s)C(t) + S(t)C(s)$, for all $s, t \in \mathbb{R}$;
- (ix) $C(t)$, $S(s)$, $C(s)$, $S(t)$ commute for $s, t \in \mathbb{R}$;
- (x) there exist constants $K \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Ke^{\omega|t|} \text{ and } |S(t) - S(\hat{t})| \leq K \left| \int_{\hat{t}}^t e^{\omega|s|} ds \right| \text{ for}$$
 all $t, \hat{t} \in \mathbb{R}$.

Throughout this paper we will make the following suppositions on A and B . Recall that $\{T(t): t \in \mathbb{R}\}$ is said to be a strongly continuous

group of linear operators on X provided $T(0) = I$, $T(t + s) = T(t)T(s)$ for all $t, s \in \mathbb{R}$, and for each $x \in X$, $T(\cdot)x: \mathbb{R} \rightarrow X$ is continuous.

(2.1) A and B are densely defined linear operators on X with domains $D(A) \subseteq D(B)$.

(2.2) $G = A + B^2$ generates a strongly continuous cosine family $\{C(t): t \in \mathbb{R}\}$.

(2.3) $-B$ generates a strongly continuous group $\{T(t): t \in \mathbb{R}\}$ of linear operators on X .

We will also refer to the regularity conditions:

(2.4) $D(G) = D(A + B^2) \subseteq D(A)$.

(2.5) $T(t): D(A) \rightarrow D(A)$.

(2.6) $E \subseteq D(B)$ and if $\{S(t): t \in \mathbb{R}\}$ is the sine family associated with $C(t)$, then $t \rightarrow BS(t)x$ is continuous for each $x \in X$ and if $x \in D(B)$ then $S(t)Bx = BS(t)x$.

(2.7) If $x \in D(B)$, then $C(t)x \in D(B)$ and $C(t)Bx = BC(t)x$.

(2.8) For $x \in X$, $\int_r^s T(u)S(u)x \, du \in D(A)$ and

$$A \int_r^s T(u)S(u)x \, du = T(s)C(s)x - T(r)C(r)x \\ + BT(s)S(s)x - BT(r)S(r)x.$$

The authors do not know if (2.8) is a consequence of (2.1) - (2.7); however, it is observed in section 5 that the examples satisfy (2.1) - (2.8).

PROPOSITION 2.2 (Travis and Webb [5]). *Suppose P is a closed linear operator on X such that*

- (i) $S(t) \in D(P)$ for all $t \in \mathbb{R}$ and $x \in X$; and
- (ii) for each $x \in X$, the map $t \rightarrow PS(t)x$ is continuous.

Then there exists $M \geq 1$ and $\omega^* \geq \omega$ such that $|PS(t)| \leq Me^{\omega^*|t|}$ for all $t \in \mathbb{R}$, where ω is given in Proposition 2.1(x).

REMARK. Assuming condition (2.6), there exists $M \geq 1$ and $\omega^* \geq \omega$ such that $|BS(t)| \leq Me^{\omega^*|t|}$.

The following proposition justifies referring to a solution of equation (1.2) as a mild solution of (1.1).

PROPOSITION 2.3. Suppose (2.1) - (2.8) hold and $g: \mathbb{R} \rightarrow X$ is continuous. If $g: \mathbb{R} \rightarrow X$ is continuously differentiable, $x \in D(G)$ with $Bx \in E$, $y \in E$, and u satisfies

$$(2.9) \quad u(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)S(t-s)g(s) ds,$$

then $u(t) \in D(A)$, $u'(t) \in D(B)$ for all $t \in \mathbb{R}$, u is twice continuously differentiable, and u satisfies

$$(2.10) \quad \begin{aligned} u''(t) + 2Bu'(t) &= Au(t) + g(t) \\ u(0) &= x, \quad u'(0) = y. \end{aligned}$$

Conversely, if u is twice continuously differentiable, $u(t) \in D(A)$ and $u'(t) \in D(B)$ for $t \in \mathbb{R}$, and u satisfies (2.10), then u satisfies (2.9).

Proof. To show that a solution of equation (2.9) satisfies (2.10), we first define

$$v(t) = \int_0^t T(t-s)S(t-s)g(s) ds.$$

Then

$$\begin{aligned}
v(t) &= \int_0^t T(t-s)S(t-s)g(0) \, ds + \int_0^t \int_u^t T(t-s)S(t-s)g'(u) \, ds \, du \\
&= \int_0^t T(t-s)S(t-s)g(0) \, ds + \int_0^t \int_0^{t-u} T(s)S(s)g'(u) \, ds \, du
\end{aligned}$$

Using condition (2.8), we have $v(t) \in D(A)$ and

$$\begin{aligned}
Av(t) &= T(t)C(t)g(0) - g(0) + BT(t)S(t)g(0) \\
&\quad + \int_0^t [T(t-u)C(t-u)g'(u) - g'(u) + BT(t-u)S(t-u)g'(u)] \, du \\
&= T(t)C(t)g(0) - g(0) + BT(t)S(t)g(0) \\
&\quad + \int_0^t [T(t-u)C(t-u)g'(u) + BT(t-u)S(t-u)g'(u)] \, du - g(t).
\end{aligned}$$

Also,

$$\begin{aligned}
v'(t) &= T(t)S(t)g(0) + \int_0^t T(s)S(s)g'(t-s) \, ds \\
&= T(t)S(t)g(0) + \int_0^t T(t-s)S(t-s)g'(s) \, ds
\end{aligned}$$

and

$$\begin{aligned}
v''(t) &= T(t)C(t)g(0) - BT(t)S(t)g(0) \\
&\quad + \int_0^t [BT(t-s)S(t-s)g'(s) + T(t-s)C(t-s)g'(s)] \, ds \\
&= Av(t) + g(t)
\end{aligned}$$

Defining $V_H(t) = T(t)[C(t)x + S(t)(Bx + y)]$ and using a straightforward computation, one can establish that

$$v_H''(t) + 2Bv_H'(t) = Av_H(t).$$

Noting that $u(t) = v_H(t) + v(t)$, it follows that u satisfies (2.10).

To establish the converse statement, observe that

$$\begin{aligned} \frac{d}{ds} T(t-s)S(t-s)u'(s) &= T(t-s)[-C(t-s)u(s) + S(t-s)u'(s)] \\ &\quad + BT(t-s)S(t-s)u'(s) \\ &= -T(t-s)C(t-s)u'(s) \\ &\quad + T(t-s)S(t-s)[Au(s) - 2Bu'(s) + g(s)] \\ &\quad + BT(t-s)S(t-s)u'(s) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{ds} T(t-s)C(t-s)u(s) &= T(t-s)[-(A + B^2)S(t-s)u(s) + C(t-s)u'(s)] \\ &\quad + BT(t-s)C(t-s)u(s) \end{aligned}$$

Integrating we obtain

$$\begin{aligned} -T(t)S(t)u(0) &= \int_0^t [-T(t-s)C(t-s)u'(s) + T(t-s)S(t-s)Au(s) \\ &\quad - T(t-s)S(t-s)Bu'(s) + T(t-s)S(t-s)g(s)] ds \end{aligned}$$

and

$$\begin{aligned} u(t) - T(t)C(t)u(0) &= \int_0^t [-T(t-s)AS(t-s)u(s) - T(t-s)B^2S(t-s)u(s) \\ &\quad + T(t-s)C(t-s)u'(s) + BT(t-s)C(t-s)u(s)] ds \end{aligned}$$

Addition of the two formulas yields

$$\begin{aligned}
u(t) &= T(t)S(t)u'(0) - T(t)C(t)u(0) \\
&= \int_0^t T(t-s)S(t-s)g(s) \, ds \\
&\quad + \int_0^t [BT(t-s)C(t-s)u(s) - T(t-s)S(t-s)Bu'(s) \\
&\quad \quad - T(t-s)B^2S(t-s)u(s)] \, ds \\
&= \int_0^t T(t-s)S(t-s)g(s) \, ds - \int_0^t \frac{d}{ds} [T(t-s)S(t-s)Bu(s)] \, ds \\
&= \int_0^t T(t-s)S(t-s)g(s) \, ds + T(t)S(t)Bu(0)
\end{aligned}$$

and it is seen that u satisfies (2.9).

3. EXISTENCE. In this section we establish the existence of solutions to equation (1.2) under various assumptions on the cosine family $C(t)$ generated by $G = A + B^2$ and the nonlinear function F .

PROPOSITION 3.1 (Fattorini [1]). *If G is the generator of a strongly continuous cosine family then there exists a translation $G_c \equiv G - c^2I$ of G such that*

- (i) G_c^{-1} exists as a bounded operator on X and
- (ii) for $0 \leq \alpha \leq 1$ the fractional powers $(-G_c)^\alpha$ exist as closed, densely defined operators with

$$D(G) \subset D((-G_c)^{\alpha_1}) \subset D((-G_c)^{\alpha_2}) \text{ for } 0 \leq \alpha_2 < \alpha_1 \leq 1.$$

The existence of $(-G_c)^{-1}$ implies that $(-G_c)^{-\alpha}$ exists as a bounded linear operator on X and consequently $D((-G_c)^\alpha)$ becomes a Banach space X_α with norm $|x|_\alpha = |(-G_c)^\alpha x|$. Also in [1], it was shown that if $X = \mathcal{L}^p$,

$1 < p < \infty$ then $E \subset D((-G_c)^{1/2})$ and for each $x \in X$, $(-G_c)^{1/2}S(\cdot)x: \mathbb{R} \rightarrow X$ is continuous. For general Banach space X , Rankin [2] showed that $E \subset D((-G_c)^\alpha)$ for all $0 \leq \alpha < 1/2$. We shall make the following assumptions:

(3.1) There exists $0 < \lambda < 1$ such that $E \subset D((-G_c)^\lambda)$ and $(-G_c)^\lambda S(\cdot)x: \mathbb{R} \rightarrow X$ is continuous for $x \in X$.

(3.2) If $0 \leq \alpha \leq 1$ and $x \in D((-G_c)^\alpha)$, then $T(t)x \in D((-G_c)^\alpha)$ with $(-G_c)^\alpha T(t)x = T(t)(-G_c)^\alpha x$ for all $t \in \mathbb{R}$.

REMARK. If condition (3.1) holds, then by Proposition 2.2 there exists $M_\lambda \geq 1$ and $\omega_\lambda \geq \omega$ such that $|(-G_c)^\lambda S(t)| \leq M_\lambda e^{\omega_\lambda |t|}$ for all $t \in \mathbb{R}$.

THEOREM 3.1. In addition to (2.1) - (2.3), (3.1), and (3.2), suppose $D \subset X_\lambda$ is open. If $F: D \rightarrow X$ satisfies $|F(x_1) - F(x_2)| \leq L|x_1 - x_2|_\lambda$ for some $L > 0$ and all $x_1, x_2 \in D$, then for each $x \in D \cap D(B)$ and $y \in X$ there exists $a > 0$ and a unique continuous function $u: [-a, a] \rightarrow X_\lambda$ such that u satisfies (1.2).

Proof. The proof employs the contraction mapping principle. Choose $\delta > 0$ and $N > 0$ such that if

$$W(x, \delta) = \{z \in X : |z - x|_\lambda < \delta\}$$

then $W(x, \delta) \subset D$ and $|F(z)| \leq N$ for $z \in W(x, \delta)$. Choose $a > 0$ such that for $t \in [-a, a]$

$$|T(t)C(t)x - x|_\lambda + |T(t)S(t)(Bx + y)|_\lambda \leq \frac{\delta}{2},$$

$$N \int_{-a}^a |T(s)(-G_c)^\lambda S(s)| ds < \frac{\delta}{2}, \text{ and}$$

$$L \int_{-a}^a |T(s)(-G_c)^\lambda S(s)| ds < 1.$$

Define

$$K = \{v \in C([-a, a]; X_\lambda) : \sup_{-a \leq t \leq a} |v(t) - x|_\lambda \leq \delta\}$$

and the map $H: K \rightarrow C([-a, a]; X_\lambda)$ by

$$[Hv](t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)S(t-s)F(v(s)) ds$$

The choice of δ and a implies that $H: K \rightarrow K$. Furthermore, for

$v_1, v_2 \in K$

$$\begin{aligned} |Hv_1(t) - Hv_2(t)|_\lambda &\leq \int_0^t |T(t-s)(-G_c)^\lambda S(t-s)[F(v_1(s)) - F(v_2(s))]| ds \\ &\leq L \int_0^t |T(t-s)(-G_c)^\lambda S(t-s)| |v_1(s) - v_2(s)|_\lambda ds \end{aligned}$$

and by the choice of a we have that H satisfies the hypothesis of the contraction mapping principle. The assertions follow.

THEOREM 3.2. *In addition to assumptions (2.1) - (2.5), (3.1), and (3.2), suppose $(-G_c)^{-1}$ is compact. Let $D \subset X_\lambda$ be open and $0 < \beta$. If $F: D \rightarrow X_\beta$ is continuous, then for each $x \in D \cap D(B)$ and $y \in X$ there exist $a > 0$ and a continuous function $u: [-a, a] \rightarrow X_\lambda$ satisfying (1.2).*

Proof. Let $\delta > 0$ and $N > 0$ be such that if

$$W(x, \delta) = \{z \in X : |z - x|_\lambda < \delta\}$$

then $W(x, \delta) \subset D$ and $|F(z)|_\beta \leq N$ for $z \in W(x, \delta)$. Choose $a > 0$ such that

$$|T(t)C(t)x - x|_\lambda + |T(t)S(t)(Bx + y)|_\lambda \leq \frac{\delta}{2}$$

and

$$N \int_{-a}^a |T(s)(-G_c)^\lambda S(s)| ds < \frac{\delta}{2}.$$

Define the set K and map H as in the proof of Theorem 3.1. As in Theorem 3.1, the choice of δ and α implies that $H: K \rightarrow K$. For $v_1, v_2 \in K$,

$$|Hv_1(t) - Hv_2(t)|_\lambda \leq \int_0^t |T(t-s)(-G_c)^\lambda S(t-s)| |F(v_1(s)) - F(v_2(s))| ds$$

and the continuity of H follows from the continuity of $F: D \rightarrow X_\beta$. To show that $\{Hv: v \in K\}$ is an equicontinuous family in $\mathcal{C}([-a, a]; X_\lambda)$, we observe that if $(G_c)^{-1}$ is compact then $(-G_c)^{-\alpha}$ is compact for $0 < \alpha < 1$ (see Travis and Webb [6]). For $-a \leq t_1 < t_2 \leq a$ and $v \in K$

$$\begin{aligned} & |Hv(t_1) - Hv(t_2)|_\lambda \\ & \leq |T(t_2)C(t_2)x - T(t_1)C(t_1)x|_\lambda + |T(t_2)S(t_2)(Bx+y) - T(t_1)S(t_1)(Bx+y)|_\lambda \\ & \quad + \int_{t_1}^{t_2} |T(t_2-s)S(t_2-s)F(v(s))|_\lambda ds \\ & \quad + \int_0^{t_1} |[T(t_2-s)S(t_2-s) - T(t_1-s)S(t_1-s)]F(v(s))|_\lambda ds. \end{aligned}$$

Now write

$$\begin{aligned} & \int_0^{t_1} |[T(t_2-s)S(t_2-s) - T(t_1-s)S(t_1-s)]F(v(s))|_\lambda ds \\ & \leq \int_0^{t_1} |T(t_2-s)[(-G_c)^\lambda S(t_2-s) - (-G_c)^\lambda S(t_1-s)](-G_c)^{-\beta}(-G_c)^\beta F(v(s))| ds \\ & \quad + \int_0^{t_1} |T(t_2-s)[T(t_2-t_1) - I](-G_c)^{-\beta}(-G_c)^\lambda S(t_1-s)(-G_c)^\beta F(v(s))| ds. \end{aligned}$$

The equicontinuity of the family $\{Hv: v \in K\}$ follows since

$$\{(-G_c)^{-\beta}(-G_c)^\beta F(y): y \in W(x, \delta)\}$$

and

$$\{(-G_c)^{-\beta}(G_c)^\lambda S(t_1 - s)(-G_c)^\beta F(y) : y \in W(x, \delta), 0 \leq s \leq t_1 \leq a\}$$

are precompact sets in X and the maps $t \rightarrow (-G_c)^\lambda S(t)$ and $t \rightarrow T(t)$ are continuous uniformly on compact sets of X . Also, for each $v \in K$ and $t \in [-a, a]$

$$\begin{aligned} (-G_c)^\lambda H v(t) &= T(t)C(t)(-G_c)^\lambda x + T(t)(-G_c)^\lambda S(t)(Bx + y) \\ &\quad + \int_0^t T(t-s)(-G_c)^{-\beta}(-G_c)^\lambda S(t-s)(-G_c)^\beta F(v(s)) ds \end{aligned}$$

and consequently, $\{Hv(t) : v \in K \text{ and } t \in [-a, a]\}$ is precompact in X_λ . Thus by the Ascoli-Arzelà Theorem $\{Hv : v \in K\}$ is precompact in X_λ and the assertions of the theorem follow from the Schauder Fixed Point Theorem.

THEOREM 3.3. *In addition to assumptions (2.1) - (2.5), (3.1), and (3.2), suppose $(-G_c)^{-1}$ is compact and $0 \leq \beta < \lambda$. If $D \subset X_\beta$ is open and $F:D \rightarrow X$ is continuous, then for each $x \in D \cap D(B)$ and $y \in X$ there exist $a > 0$ and a continuous function $u: [-a, a] \rightarrow X_\beta$ satisfying (1.2).*

Proof. The proof is similar to that of the previous theorem. Let $\delta > 0$ and $N > 0$ be such that if

$$W(x, \delta) = \{z \in X : |z - x|_\beta < \delta\}$$

then $W(x, \delta) \subset D$ and $|F(z)| \leq N$ for $z \in W(x, \delta)$. Define K and H as before. One shows $H: K \rightarrow K$ and is continuous. Writing

$$\begin{aligned} (-G_c)^\beta H v(t) &= T(t)C(t)(-G_c)^\beta x + T(t)(-G_c)^\beta S(t)(Bx + y) \\ &\quad + \int_0^t T(t-s)(-G_c)^{\beta-\lambda}(-G_c)^\lambda S(t-s)F(v(s)) ds \end{aligned}$$

one observes that $\{Hv(t): v \in K \text{ and } t \in [-a, a]\}$ is precompact in X_β and writing

$$\begin{aligned} & |Hv(t_2) - Hv(t_1)|_\beta \\ & \leq |T(t_2)C(t_2)x - T(t_1)C(t_1)x|_\beta + |T(t_2)S(t_2)(Bx + y) - T(t_1)S(t_1)(Bx + y)|_\beta \\ & \quad + \int_{t_1}^{t_2} |T(t_2 - s)S(t_2 - s)F(v(s))|_\beta ds \\ & \quad + \int_0^{t_1} |T(t_2 - s)[(-G_c)^\lambda S(t_2 - s) - (-G_c)^\lambda S(t_1 - s)](-G_c)^{\beta-\lambda}F(v(s))| ds \\ & \quad + \int_0^{t_1} |T(t_2 - s)[T(t_2 - t_1) - I](-G_c)^{\beta-\lambda}(G_c)^\lambda S(t_1 - s)F(v(s))| ds \end{aligned}$$

one observes that the family $\{Hv: v \in K\}$ is equicontinuous. Consequently, $H(K) \subset X$ is precompact the the assertions follow.

REMARK. In addition to the hypothesis of Theorem 3.1, 3.2, or 3.3, suppose the regularity conditions (2.6) and (2.7) hold. Then if $x \in E$ and $y \in X$, we have that u' exists and $u': [-a, a] \rightarrow X$ is continuous. If the regularity conditions (2.4) - (2.8) hold, F is continuously Frechet differentiable, $x \in D(G)$ with $Bx \in E$, and $y \in E$, then u satisfies (1.1).

The final theorem of this section gives sufficient criteria for global existence.

THEOREM 3.4. Assume that either

- (i) the suppositions of Theorem 3.1 hold, or
- (ii) the suppositions of Theorem 3.2 (3.3) hold and F maps bounded sets of D into bounded sets of $X_\beta(X)$.

and the regularity conditions (2.6) - (2.7) hold. If $x \in E$ and $y \in X$ and u is a solution of equation (1.2) noncontinuable to the right on $[0, d]$, then either $d = +\infty$ or, given any closed bounded set $V \subset D$, there exists a sequence $t_k \rightarrow d^-$ such that $u(t_k) \notin V$. An analogous result holds for noncontinuity to the left.

Proof. The proofs under assumptions (i) and (ii) are similar; only assumption (ii) with Theorem 3.2 is considered. For contradiction, suppose $d < \infty$ and there exists a bounded closed set $V \subset D$ such that $u(t) \in V$ for all $t \in [0, d]$. For $0 \leq t_1 < t_2 < d$ and u satisfying equation (1.2), we have

$$\begin{aligned} |u(t_2) - u(t_1)|_\lambda &\leq |T(t_2)C(t_2)(-G_c)^\lambda x - T(t_1)C(t_1)(-G_c)^\lambda x| \\ &\quad + |[T(t_2)(-G_c)^\lambda S(t_2) - T(t_1)(-G_c)^\lambda S(t_1)](Bx + y)| \\ &\quad + \int_{t_1}^{t_2} |T(t_2 - s)(-G_c)^\lambda S(t_2 - s)F(u(s))| ds \\ &\quad + \int_0^{t_1} |[T(t_2 - s)(-G_c)^\lambda S(t_2 - s) \\ &\quad \quad - T(t_1 - s)(-G_c)^\lambda S(t_1 - s)]F(u(s))| ds. \end{aligned}$$

Noting that $\{F(u(s)): 0 \leq s < d\} \subset F(V)$ is bounded in X_β and thus pre-compact in X and that $t \rightarrow T(t)(-G_c)^\lambda S(t)$ is uniformly continuous on compact sets in X we see that $\lim_{t \rightarrow d^-} u(t)$ exists with $\lim_{t \rightarrow d^-} u(t) = p \in V \subset D$. Also,

$$\begin{aligned}
& |u'(t_2) - u'(t_1)| \\
& \leq |T(t_2)GS(t_2)x - T(t_1)GS(t_1)x| + |T(t_2)BC(t_2)x - T(t_1)BC(t_1)x| \\
& \quad + |T(t_2)C(t_2)(Bx + y) - T(t_1)C(t_1)(Bx + y)| \\
& \quad + |T(t_2)BS(t_2)(Bx + y) - T(t_1)BS(t_1)(Bx + y)| \\
& \quad + \int_{t_1}^{t_2} |T(t_2 - s)C(t_2 - s)F(u(s)) - T(t_2 - s)BS(t_2 - s)F(u(s))| ds \\
& \quad + \int_0^{t_1} |[T(t_2 - s)C(t_2 - s) - T(t_1 - s)C(t_1 - s)]F(u(s))| ds \\
& \quad + \int_0^{t_1} |[T(t_2 - s)BS(t_2 - s) - T(t_1 - s)BS(t_1 - s)]F(u(s))| ds.
\end{aligned}$$

Again, using the fact that $\{F(u(s)): 0 \leq s < d\}$ is precompact in X and that $t \rightarrow T(t)C(t)$ and $t \rightarrow T(t)BS(t)$ are continuous uniformly on compact sets of X , it follows that $\lim_{t \rightarrow d^-} u'(t) = q \in X$ exists. Noting that

$$p = T(d)[C(d)x + S(d)(Bx + y)] + \int_0^d T(d - s)S(d - s)F(u(s)) ds$$

and

$$\begin{aligned}
q = & T(d)[GS(d)x + C(d)(Bx + y)] - BT(d)[C(d)x + S(d)(Bx + y)] \\
& - \int_0^d BT(d - s)S(d - s)F(u(s)) ds + \int_0^d T(d - s)C(d - s)F(u(s)) ds,
\end{aligned}$$

we see that $p \in D \cap D(B)$ and $q \in X$. Thus one can obtain a solution of the equation

$$v(t) = T(t - d)[C(t - d)p + S(t - d)(Bp + q)] + \int_d^t T(t - s)S(t - s)F(v(s)) ds$$

for $d \leq t < d^*$. One then extends u to $[0, d^*)$ by defining $u(t) = v(t)$ on $[d, d^*)$. Using the properties found in Proposition 2.1 (in particular, identities (vii) and (viii)), one can show for $t \in [d, d^*)$ that

$$u(t) = v(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s) S(t-s) F(u(s)) ds,$$

contradicting the noncontinuability of u .

4. ASYMPTOTIC BEHAVIOR. In this section we will assume $T(t)$ decays exponentially as $t \rightarrow \infty$; i.e., there exists $b > 0$ such that $|T(t)| \leq Me^{-bt}$ for all $t \in [0, \infty)$. For convenience, we assume $M = 1$.

THEOREM 4.1. *In addition to the suppositions of Theorem 3.2, suppose F maps bounded sets of X_λ into bounded sets of X_β . Also, suppose $b < \omega_\lambda$, $x \in X_\gamma$ for some $\lambda < \gamma < 1$ with $Bx \in X_\beta$, $y \in X_\beta$ and u is a solution of (1.2) defined and bounded on $[0, \infty)$. Then $\{u(t) : t \geq 0\}$ is precompact in X_λ . A similar assertion holds under the hypotheses of Theorem 3.3.*

Proof. Choose $1 > \gamma > 0$ such that $\lambda < \gamma < \lambda + \beta$. Then

$$\begin{aligned} (-G_c)^\gamma u(t) &= (-G_c)^\gamma T(t)[C(t)x + S(t)(Bx + y)] \\ &\quad + (-G_c)^\gamma \int_0^t T(t-s) S(t-s) F(u(s)) ds \\ &= T(t)C(t)(-G_c)^\gamma x + T(t)(-G_c)^\lambda S(t)[(-G_c)^{\gamma-\lambda}(Bx + y)] \\ &\quad + \int_0^t T(t-s)(-G_c)^\lambda S(t-s)(-G_c)^{\gamma-\lambda} F(u(s)) ds \end{aligned}$$

and thus

$$|(-G_c)^Y u(t)| \leq K e^{(-b+\omega)t} |x|_Y + M_\lambda e^{(-b+\omega_\lambda)t} |Bx + y|_{Y-\lambda} \\ + M_\lambda \int_0^t e^{(-b+\omega_\lambda)(t-s)} |F(u(s))|_{Y-\lambda} ds.$$

Consequently, $\{(-G_c)^Y u(t) \mid$ is bounded and $\{(-G_c)^\lambda u(t) : t \geq 0\} = \{(-G_c)^{\lambda-Y} (-G_c)^Y u(t) : t \geq 0\}$ is precompact in X .

THEOREM 4.2. *In addition to the hypotheses of Theorem 3.4 with $D = X$, suppose there exists a continuous function $j: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $j(0) = 0$ such that $|F(x)|_\beta \leq j(r) |x|_\lambda$ for $x \in X_\lambda$ and $|x|_\lambda \leq r$. If $x \in X_\lambda \cap D(B)$ and $y \in X$, then there exists $\varepsilon > 0$, $N \geq 1$, and $\delta > 0$ such that if $|x|_\lambda \leq \varepsilon/2$ and $|y| \leq \varepsilon/2$ then the solution of (1.2) exists on $[0, \infty)$ and satisfies $|u(t)|_\lambda \leq N e^{-\delta t} (|x|_\lambda + |Bx + y|)$.*

Proof. Let $N = \max\{K, M_\lambda\}$, $\varepsilon_1 > 0$ such that $j(\varepsilon_1) < (b - \omega_\lambda)/2$, and $\varepsilon = \varepsilon_1/N$. For $|x|_\lambda \leq \varepsilon/2$ and $|y| \leq \varepsilon/2$, let u be the solution of equation (1.2) and $[0, t^*]$ ($[0, \infty)$ if $t^* = \infty$) the maximal interval such that $|u(t)|_\lambda \leq \varepsilon_1$ for all $0 \leq t < t^*$. For $0 \leq t < t^*$,

$$(-G_c)^\lambda u(t) \\ = T(t) [C(t) (-G_c)^\lambda x + (-G_c)^\lambda S(t) (Bx + y)] + \int_0^t T(t-s) (-G_c)^\lambda S(t-s) F(u(s)) ds$$

and

$$|u(t)|_\lambda \\ \leq e^{-bt} [K e^{\omega t} |x|_\lambda + M_\lambda e^{\omega_\lambda t} |Bx + y|] + \left[\frac{b - \omega_\lambda}{2} \right] M_\lambda \int_0^t e^{-b(t-s)} e^{\omega_\lambda(t-s)} |u(s)|_\lambda ds.$$

Thus

$$e^{(b-\omega_\lambda)t} |u(t)|_\lambda \leq K|x|_\lambda + M_\lambda |Bx + y| + \frac{b-\omega_\lambda}{2} M_\lambda \int_0^t e^{(b-\omega_\lambda)s} |u(s)|_\lambda ds$$

and Gronwall's inequality yields

$$|u(t)|_\lambda \leq N(|x|_\lambda + |Bx + y|) e^{[(b-\omega_\lambda)/2](Nt)}.$$

Thus $\delta = [(b - \omega_\lambda)/2]N$ and $t^* = \infty$ by Theorem 3.4.

5. EXAMPLES. We first consider the equation

$$(5.1) \quad w_{tt}(x,t) + 2b(x)w(x,t) = w_{xx}(x,t) + f(w(x,t)), \quad 0 < x < \pi, t \in \mathbb{R}$$

$$w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \leq x \leq \pi$$

where h and g are in $\mathcal{L}_2(0,\pi;\mathbb{R})$ and $b: [0,\pi] \rightarrow \mathbb{R}$ is continuous.

Let $X = \mathcal{L}_2(0,\pi;\mathbb{R})$ with inner product (\cdot, \cdot) and define $A: D(A) \rightarrow X$ by $A\phi = \phi''$ where

$$D(A) = \{\phi \in X: \phi, \phi' \text{ are absolutely continuous, } \phi'' \in X, \phi(0) = \phi(\pi) = 0\}.$$

A can be written in the form

$$A\phi = - \sum_{n=1}^{\infty} n^2 (\phi, \phi_n) \phi_n$$

for $\phi \in D(A)$, where $\phi_n(x) = (2/\pi)^{1/2} \sin(nx)$. We define $B: X \rightarrow X$ by $[B\phi]x = b(x)\phi(x)$. Defining $G = A + B^2$ we have

$$[G\phi]x = \sum_{n=1}^{\infty} (-n^2 + b(x)) (\phi, \phi_n) \phi_n(x)$$

and G generates the cosine family

$$[C(t)\phi](x) = \sum_{n=1}^{\infty} C_n(x, t) (\phi, \phi_n) \phi_n(x)$$

where

$$C_n(x, t) = \begin{cases} \cos(n^2 - b^2(x))^{1/2} t & , \quad n^2 > b^2(x) \\ 1 & , \quad n^2 = b^2(x) \\ \cosh(b^2(x) - n^2)^{1/2} t & , \quad n^2 < b^2(x). \end{cases}$$

Also,

$$[S(t)f](x) = \sum_{n=1}^{\infty} s_n(x, t) (\phi, \phi_n) \phi_n(x)$$

where

$$s_n(x, t) = \begin{cases} (n^2 - b^2(x))^{-1/2} \sin(n^2 - b^2(x))^{1/2} t & , \quad n^2 > b^2(x) \\ t & , \quad n^2 = b^2(x) \\ (b^2(x) - n^2)^{-1/2} \sinh(b^2(x) - n^2)^{1/2} t & , \quad n^2 < b^2(x). \end{cases}$$

Note also that $-B$ generates the group $\{T(t): t \in \mathbb{R}\}$ on X defined by $[T(t)\phi]x = e^{-tb(x)} \phi(x)$ and $D(B) = X$. It is easily seen that properties (2.1) - (2.7) are satisfied. Property (2.8) is established by the following proposition.

PROPOSITION 5.1. *Let $A, B, C(t), S(t), T(t)$ be as above. Then (2.8) is satisfied, i.e., for $\phi \in X$, $\int_T^S T(u)S(u)\phi \, du \in D(A)$ and*

$$A \int_T^S T(u)S(u)\phi \, du = T(s)C(s)\phi - T(r)C(r)\phi + BT(s)S(s)\phi - BT(r)S(r)\phi.$$

Proof. Using the fact that $C(t)T(s) = T(s)C(t)$ for all $t, s \in \mathbb{R}$ and identity (viii) in Proposition (2.1), we have

$$j(t) \stackrel{\text{def}}{=} C(t) \int_r^s T(u)S(u)\phi \, du = \frac{1}{2} \int_r^s T(u)(S(u+t) + S(u-t))\phi \, du.$$

Thus

$$\begin{aligned} j'(t) &= \frac{1}{2} \int_r^s T(u)(C(u+t) - C(u-t))\phi \, du \\ &= \frac{1}{2} \int_{r+t}^{s+t} T(u-t)C(u)\phi \, du - \frac{1}{2} \int_{r-t}^{s-t} T(u+t)C(u)\phi \, du \end{aligned}$$

and

$$\begin{aligned} j''(t) &= \frac{1}{2} [T(s)C(s+t)\phi - T(r)C(r)\phi] + \frac{1}{2} \int_{r+t}^{s+t} BT(u-t)C(u)\phi \, du \\ &\quad + \frac{1}{2} [T(s)C(s-t)\phi - T(r)C(r-t)\phi] + \frac{1}{2} \int_{r-t}^{s-t} BT(u+t)C(u)\phi \, du. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} (A + B^2) \int_r^s T(u)S(u)\phi \, du &= C''(0) \int_r^s T(u)S(u)\phi \, du \\ &= T(s)C(s)\phi - T(r)C(r)\phi + \int_r^s BT(u)C(u)\phi \, du \\ &= T(s)C(s)\phi - T(r)C(r)\phi + BT(s)S(s)\phi - BT(r)S(r)\phi \\ &\quad + \int_r^s B^2T(u)C(u)\phi \, du \end{aligned}$$

from which (2.8) follows.

As noted in the comments preceding condition (3.1), condition (3.1) is satisfied with $\lambda = 1/2$. Also, if c is such that $b^2(x) - c^2 \leq 0$

for all $x \in [0, \pi]$, we see that G_c^{-1} exists, satisfies

$$[G_c^{-1}\phi](x) = \sum_{n=1}^{\infty} (-n^2 + b^2(x) - c^2)^{-1} (\phi, \phi_n) \phi_n(x),$$

and is compact. Furthermore,

$$[(-G_c)^{1/2}\phi]x = \sum_{n=1}^{\infty} (n^2 - b^2(x) + c^2)^{1/2} (\phi, \phi_n) \phi_n(x).$$

Since

$$\begin{aligned} D((-G_c)^{1/2}) &= \{\phi \in X: \sum_{n=1}^{\infty} (n^2 - b(x) + c^2) (\phi, \phi_n)^2 < \infty\} \\ &= \{\phi \in X: \sum_{n=1}^{\infty} n^2 (\phi, \phi_n)^2 < \infty\}, \end{aligned}$$

we have from Travis and Webb [7] that

$$\begin{aligned} D((-G_c)^{1/2}) &= \{\phi \in X: \phi \text{ is absolutely continuous,} \\ &\quad \phi' \in X, \text{ and } \phi(0) = \phi(\pi) = 0\}. \end{aligned}$$

Noting that for $\phi \in D((-G_c)^{1/2})$

$$\begin{aligned} |\phi'| &= \int_0^\pi \left[\sum_{n=1}^{\infty} (\phi', \phi_n) \phi_n(x) \right]^2 dx \\ &= \int_0^\pi \left[\sum_{n=1}^{\infty} n \left(\frac{2}{\pi} \right)^{1/2} (\phi, \phi_n) \cos(nx) \right]^2 dx \\ &= \sum_{n=1}^{\infty} n^2 (\phi, \phi_n)^2 \end{aligned}$$

and

$$\begin{aligned}
|\phi|_{1/2} &= |-G_c^{1/2} \phi| \\
&= \int_0^\pi \left[\sum_{n=1}^\infty (n^2 - b^2(x) + c^2)^{1/2} (\phi, \phi_n) \phi_n(x) \right]^2 dx \\
&= \sum_{n=1}^\infty (n^2 - b^2(x) + c^2) (\phi, \phi_n)^2
\end{aligned}$$

it follows that there exists $K > 0$ such that for all $\phi \in D((-G_c)^{1/2})$

$$K|\phi'| \geq |\phi|_{1/2} \geq |\phi'|.$$

If $f(w) = -aw - bw^3$, $a, b > 0$ or $f(w) = \sin w$ then equation (5.1) is the Klein-Gordon or Sine-Gordon equation respectively and $[F(\phi)]x = f(\phi(x))$ satisfies the conditions of Theorem 3.2 with $F: X_{1/2} \rightarrow X_{1/2}$. For example, to see $F: X_{1/2} \rightarrow X_{1/2}$ defined by $[F(\phi)]x = -a\phi(x) - b\phi^3(x)$ is continuous, observe for $\phi, \psi \in X_{1/2}$

$$\begin{aligned}
|F\phi - F\psi|_{1/2} &\leq K|[F\phi]' - [F\psi]'| \\
&\leq K|a(\phi' - \psi') + 3b(\psi^2\psi' - \phi^2\phi')| \\
&\leq aK|\phi' - \psi'| + 3bK(|\psi^2\psi' - \psi^2\phi'| + |\phi'(\psi^2 - \phi^2)|)
\end{aligned}$$

Supp Suppose $b(x) \geq 0$ for all $x \in [0, \pi]$ and let $b_m = \min\{b(x): 0 \leq x \leq \pi\}$ and $b_M = \max\{b(x): 0 \leq x \leq \pi\}$. Then $|T(t)| \leq e^{-b_m t}$ and there exists $M_{1/2} > 0$ such that $|(-G_c)^{1/2}S(t)| \leq M_{1/2}e^{w_{1/2}t}$ where $w_{1/2} = 0$ if $b_M \leq 1$ and $w_{1/2} = (b_M^2 - 1)^{1/2}$ if $b_M > 1$. Consequently, the results of section 4 apply provided $0 < b_m < b_M \leq 1$ or $b_M > 1$ and $b_m > (b_M^2 - 1)^{1/2}$.

As another example, consider

$$(5.2) \quad w_{tt}(x,t) + 2w_{xt}(x,t) = w_{xx}(x,t) + f(w(x,t)), \quad 0 < x < \pi, \quad t \in \mathbb{R}$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \leq x \leq \pi$$

$$w(0,t) = w(\pi,t), \quad w_x(0,t) = w_x(\pi,t), \quad t \in \mathbb{R}$$

where $h, g \in \mathcal{L}_2(0, \pi; \mathbb{R})$. Again let $X = \mathcal{L}_2(0, \pi; \mathbb{R})$ with inner product (\cdot, \cdot) and define $A: D(A) \rightarrow X$ by $A\phi = \phi''$ where

$$D(A) = \{\phi \in X: \phi, \phi' \text{ are absolutely continuous,} \\ \phi'' \in X, \phi(0) = \phi(\pi), \phi'(0) = \phi'(\pi)\}.$$

A can be written in the form

$$A\phi = \sum_{n=1}^{\infty} -4n^2 [(\phi, \phi_n)\phi_n + (\phi, \psi_n)\psi_n]$$

where $\phi_n(x) = (2/\pi)^{1/2} \sin(2nx)$, $\psi_n(x) = (2/\pi)^{1/2} \cos(2nx)$, and $\phi = a + \sum_{n=1}^{\infty} (\phi, \phi_n)\phi_n + (\phi, \psi_n)\psi_n$, $a = (1/\pi) \int_0^\pi \phi(x) dx$. We define $B: D(B) \rightarrow X$ by $B\phi = \phi'$ where

$$D(B) = \{\phi \in X: \phi' \in X, \phi(0) = \phi(\pi)\}.$$

The group $\{T(t): t \in \mathbb{R}\}$ generated by $-B$ is defined by

$$[T(t)\phi]x = a + \sum_{n=1}^{\infty} (\phi, \phi_n)\phi_n(x-t) + (\phi, \psi_n)\psi_n(x-t)$$

and $G = A + B^2 = 2A$ generates the cosine family

$$[C(t)\phi]x = a + \sum_{n=1}^{\infty} \cos(2\sqrt{2}nt) [(\phi, \phi_n)\phi_n(x) + (\phi, \psi_n)\psi_n(x)].$$

Conditions (2.1) - (2.7) are satisfied and condition (2.8) is verified in the manner indicated by the proof of Proposition 5.1. Also, G^{-1} exists as a compact operator on X and $(-G_c)^{1/2}$ exists with

$$(-G_c)^{1/2}\phi = \sum_{n=1}^{\infty} 2\sqrt{2} n[(\phi, \phi_n)\phi_n + (\phi, \psi_n)\psi_n].$$

The existence results of section 3 apply to example (5.2); however, note that since $T(t)$ is of type $b = 0$ and $C(t)$ is of type $w = 0$, the asymptotic results of section 4 do not apply.

The abstract theory also applies to the equation

$$(5.3) \quad w_{tt} + 2b(x)w_t = -w_{xxxx} + f(w, w_x), \quad 0 < x < \pi, \quad t \in \mathbb{R}$$

$$w(0, t) = w(\pi, t) = w_{xx}(0, t) = w_{xx}(\pi, t) = 0, \quad t \in \mathbb{R}$$

$$w(x, 0) = g(x), \quad w_t(x, 0) = h(x), \quad 0 \leq x \leq \pi.$$

As before let $X = \mathcal{L}_2(0, \pi; \mathbb{R})$, $A\phi = -\phi''''$ with

$$D(A) = \{\phi \in X: \phi, \phi', \phi'', \phi''' \text{ are absolutely continuous,} \\ \phi''' \in X, \phi(0) = \phi(\pi) = \phi''(0) = \phi''(\pi) = 0\}.$$

In this case, $[F\phi]x = f(\phi(x), \phi'(x))$ with appropriate conditions on f satisfies $F: X_{1/4} \rightarrow X$ continuous and consequently Theorem 3.3 applies.

REMARK. The techniques also apply to

$$(5.4) \quad w_{tt} - 2w_{xxt} = -w_{xxxx} + f(w, w_x), \quad 0 < x < \pi, \quad t \geq 0$$

with the side conditions of (5.3). Here $B\phi = -\phi''$ with

$$D(B) = \{\phi \in X: \phi, \phi' \text{ are absolutely continuous,} \\ \phi'' \in X, \phi(0) = \phi(\pi) = 0\}.$$

Thus $-B$ generates an analytic semigroup. Since $A + B^2 = 0$, $C(t)x = x$, $S(t)x = tx$, and equation (1.2) has the form

$$u(t) = T(t)[x + t(Bx + y)] + \int_0^t (t-s)T(t-s)F(u(s)) ds.$$

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20. Abstract (continued)

where A and B are densely defined linear operators and F is possibly nonlinear and unbounded. Assuming that $A + B^2$ generates a cosine family $C(t)$ and $-B$ generates a group $T(t)$, there is a variation of constants formula for (1); namely

$$(2) \quad u(t) = T(t)[C(t)x + S(t)(Bx + y)$$

$$+ \int_0^t T(t-s)S(t-s)F(u(s)) \, ds,$$

where $S(t)$ is the sine family associated with $C(t)$. The motivating examples include $w_{tt} + 2b(x)w_t = w_{xx} + f(w, w_x, w_t)$ and $w_{tt} + 2w_{tx} = w_{xx} + f(w, w_x, w_t)$, for $0 < x < \pi$, $t \in \mathbb{R}$, $w(x, 0) = h(x)$, $w_t(x, 0) = g(x)$, and various boundary conditions. We examine the existence of mild solutions and the asymptotic behavior when there is a damping effect introduced by the $2Bu'(t)$ term.

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